Sliding on the Surface of a Rough Sphere

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A well-known textbook problem treats the motion of a particle sliding frictionlessly on the surface of a sphere. An interesting variation is to consider what happens when kinetic friction is present.[1] This problem can be solved exactly.

A free-body diagram is sketched in Fig. 1 and defines many of the relevant variables. The radial component of Newton's second law is

$$mg \cos \theta - N = ma_\text{c}. \tag{1}$$

Solving for N, which will be needed to compute the frictional force, we obtain

$$N = m\left(g\cos\theta - \frac{v^2}{r}\right), \tag{2}$$

where $v(\theta)$ is the speed of the object. Now we insert this into the tangential component of Newton's second law,

$$mg \sin \theta - \mu N = ma_\text{t}, \tag{3}$$

to obtain
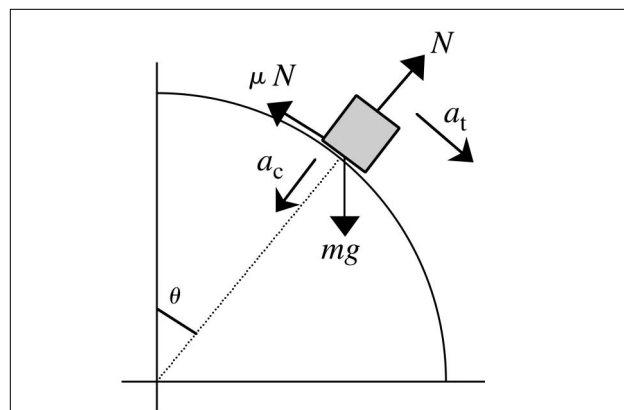
$$g(\sin\theta - \mu\cos\theta) + \frac{\mu v^2}{r} = \frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}. \tag{4}$$

Substituting for the angular speed, $d\theta/dt = v/r$, and using the identity $2v\,dv/d\theta = d(v^2)/d\theta$ we get

$$\frac{d(V^2)}{d\theta} - 2\mu V^2 = 2(\sin\theta - \mu\cos\theta), \tag{5}$$

where the dimensionless speed is $V \equiv v/\sqrt{rg}$.

A numerical solution to this differential equation can be readily obtained using finite-difference itera-

Fig. 1. A block of mass *m* sliding on the surface of a rough sphere of radius *r*, at the instant it is located at angle θ with respect to the vertical. The three forces acting on the object are the normal force *N*, gravity *mg*, and sliding friction *μN*, where μ is the coefficient of kinetic friction. The acceleration has been resolved into centripetal a_c and tangential a_t components.

tion in a spreadsheet for a given initial speed and angle.[2] The equation also can be solved analytically in a calculus-based course as explained in Appendix A. This solution is graphed in Fig. 2 for various values of the two parameters μ and V_0, where V_0 is the initial dimensionless speed at the top of the sphere. For reference, curve 1 plots the standard frictionless example of $\mu = 0$ and $V_0 = 0$, showing that the particle flies off the sphere at $\theta = \cos^{-1}(2/3) = 48.2°$ with a dimensionless speed of $V = \sqrt{2/3} = 0.816$.

There are two possible fates of the object. It will either be brought to rest on the surface or it will eventually lose contact with the sphere. The first situation occurs if there exists a value[3] of θ between 0° and 90°

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for which $V = 0$, i.e., if a plot of V versus θ hits the horizontal axis in Fig. 2. Curve 2 shows a typical example of this behavior. On the other hand, the object flies off the sphere if $N = 0$. According to Eq. (2), this happens when $V^2 = \cos \theta$, i.e., if a plot of V versus θ strays into the shaded region of the graph in Fig. 2. (In particular, V_0 is constrained to be less than 1 if the object is to even begin on the sphere. This gives physical significance to the speed $\sqrt{rg}$ used to normalize V.) For example, curve 3 shows friction initially slowing down the particle but the increasing gradient of the surface subsequently re-accelerating the mass (which occurs in general whenever it does not come to rest first, i.e., provided μ is not too large). By starting with just the right speed for a given value of μ, the particle can be slowed down arbitrarily close to zero, so that the particle appears to "bounce" off the horizontal axis in Fig. 2. For example, see curve 4, discussed in greater detail in Appendix B.

The motion of the block on the surface of the ball can now be understood by considering its trajectory in the V-versus-θ parameter space of Fig. 2. A curve begins at a point on the vertical axis at which $V = V_0$ with $0 < V_0 < 1$. The plot proceeds rightward until it ends when it contacts the horizontal axis along the bottom or the gray region limiting its upward range, whichever occurs first. It is an instructive exercise (cf. Appendix B) to prove that it is impossible for the trajectory to pass through the intersection point of these two bounding curves, i.e., the particle can never reach the equator. However, it can get arbitrarily close to 90° by skirting the shaded region all the way along (cf. curve 5 in Fig. 2).

In summary, a rich variety of curves of $v(\theta)$ are possible for a point particle sliding on the surface of a rough sphere. Generating and graphing these curves can therefore prove a profitable method for students to learn how to use a spreadsheet in introductory physics. In particular, it can help them appreciate that while it is helpful to try random values of the plot parameters (to generate curves 2 and 3 in Fig. 2, for instance), a more focused approach is necessary to sample the full range of possible trajectories (e.g., curves 4 and 5) of a nontrivial dynamical system.

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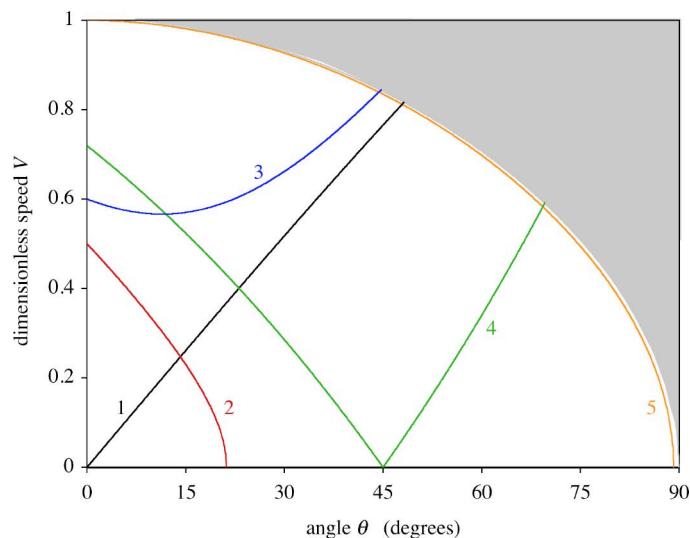
Fig. 2. Five trajectories of the particle in *V*-vs-θ space using the following parameters:

1: $\mu = 0$ and $V_0 = 0$;
2: $\mu = 0.6$ and $V_0 = 0.5$;
3: $\mu = 0.3$ and $V_0 = 0.6$;
4: $\mu = 1$ and $V_0 \cong 0.71944$ from Eq. (B4);
5: $\mu = 100$ and $V_0 \cong 0.9999625$ from Eq. (B1).

Appendix A: Analytic Solution of Eq. (5)

This linear, inhomogeneous, first-order ordinary differential equation can be directly solved for V^2 using an integrating factor.[4] But an alternative approach can be used with students who have not yet taken a differential equations course. One solution can be found by substituting the trial form $V_1^2 = A \cos \theta + B \sin \theta$ into Eq. (5) and separately equating cosine and sine terms to find

$$V_1^2 = \frac{(4\mu^2 - 2)\cos \theta - 6\mu \sin \theta}{1 + 4\mu^2}. \qquad (A1)$$

On the other hand, consider the simpler equation obtained by setting the right-hand side of Eq. (5) to zero. This can be rearranged into $d(V^2)/(V^2) = 2\mu d\theta$ and both sides integrated to obtain

$$V_2^2 = Ce^{2\mu\theta}, \qquad (A2)$$

where C is an arbitrary constant of integration. It is not hard to see that $V_1^2 + V_2^2$ is also a solution of Eq. (5). (Since this sum contains one undetermined con-

stant and the original differential equation is first order, this is in fact the general solution.[5]) Finally, the value of C is obtained by fitting this general solution to the initial conditions: The particle begins at angle θ_0 with dimensionless speed V_0. In particular, suppose that $\theta_0 = 0$, i.e., the object starts at the top of the sphere just as in the conventional frictionless problem. Unlike this standard case, however, V_0 cannot be zero because friction would then cause the particle to remain in stable equilibrium (rather than slipping out of unstable equilibrium) at the north pole of the sphere.[6]   The final solution therefore becomes

$$V^2 = \frac{(2 - 4\mu^2)(e^{2\mu\theta} - \cos\theta) - 6\mu\sin\theta}{1 + 4\mu^2} + V_0^2\, e^{2\mu\theta}.$$

(A3)

Appendix B: Approaching the Equator

Setting $V = 0$ at $\theta = \pi/2$, Eq. (A3) can be solved for V_0^2 to obtain

$$V_0^2 = \frac{4\mu^2 - 2 + 6\mu e^{-\mu\pi}}{1 + 4\mu^2}.$$

(B1)

(In order for this to be positive, μ must be larger than approximately 0.6034.) Substitute this back into Eq. (A3) and put $\theta = \pi/2 - \delta$ to obtain

$$V^2 = \frac{(4\mu^2 - 2)\sin\delta + 6\mu(e^{-\mu\delta} - \cos\delta)}{1 + 4\mu^2}.$$

(B2)

By design, this is zero at $\delta = 0$. However, it equals $(3\mu\delta - 2)\delta$ when it is expanded to second order in δ, i.e., V^2 is negative when δ is infinitesimally small. This means the particle *cannot* reach the equator: It must stop at some smaller angle.[3]

But how close to the equator can one get? In order to maximize the final angle θ, we want $d\theta/d(V^2) \to 0$ at $V = 0$. Consequently $\mu \to \infty$ according to Eq. (5). In this limit, Eq. (B2) becomes

$$V^2 = \sin\delta - \frac{3}{2\mu}\cos\delta.$$

(B3)

Therefore, the object comes to rest near $\pi/2 - 1.5/\mu$. For example, curve 5 shows what happens when $\mu = 100$. The particle actually stops at 89.2°, in good agreement with this prediction.

This is an unrealistically large value for μ. The fi-

nal angle θ is maximized for a more reasonable value of $\mu = 1$ when V_0 is chosen so that the trajectory of the particle in Fig. 2 just grazes the horizontal axis [so that both $V \to 0$ and $d(V^2)/d\theta = 0$] at, say, angle θ_r and then speeds back up and flies off the sphere at θ. The left-hand side of Eq. (5) is zero, thus implying that $\theta_r = 45°$, and Eq. (A3) can then be solved at this angle to deduce that

$$V_0^2 = 0.4\left(1 + \sqrt{2}\, e^{-\pi/2}\right).$$

(B4)

Subsequently the particle leaves the surface at angle $\theta = 69.6°$ with a dimensionless speed of $V = 0.59$, plotted as curve 4 in Fig. 2.

References

1. H. Sarafian, "How far down can you slide on a rough ball?" *AAPT Announcer* **31**, 113 (Winter 2001). Sarafian has analyzed this problem using Mathematica for a special issue of the *Journal of Symbolic Computation* to be published in late fall of 2003.

2. Many introductory textbooks now include an overview of Euler's method of numerical integration in a spreadsheet such as Excel. For example, see P.A. Tipler and G. Mosca, *Physics for Scientists and Engineers*, 5th ed. (Freeman, New York, 2003), Sec. 5-4.

3. For many values of μ and V_0, there are two mathematical solutions of θ for which $V = 0$. Only the smaller solution has physical significance for $\theta_0 = 0$.

4. Integrating factors are discussed in standard differential equation texts, such as D.G. Zill and M.R. Cullen, *Differential Equations with Boundary-Value Problems*, 3rd ed. (PWS-Kent, Boston, 1993), Sec. 2.5.

5. Students who have taken an introductory course in differential equations will recognize Eq. (A1) as a particular solution and Eq. (A2) as the complementary solution of the homogeneous equation corresponding to Eq. (5). If even this alternative approach (without the technical terminology) is too advanced, students could still be challenged to verify that Eq. (A3) satisfies both Eq. (5) and the initial conditions.

6. Another reasonable choice of initial conditions has been adopted in W. Herreman and H. Pottel, "Problem: The sliding of a mass down the surface of a solid sphere," *Am. J. Phys.* **56**, 351 (April 1988).

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